Indian Statistical Institute

Functional Analysis. Back-paper examination

M.Math First Year. 2025-2026

Time : 3 Hours   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) All spaces are considered over $\mathbb{C}$.

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(1) (15 marks) Let X be a normed space, and let $x \in X$. Prove that

$$\|x\| = \sup\{|\varphi(x)| : \varphi \in X^*, \|\varphi\| \leq 1\}.$$

(2) (15 marks) Prove that $C[0,1]$ is not complete with respect to the $\|\cdot\|_p$-norm for any $1 \leq p < \infty$.

(3) (15 marks) Let T be a compact operator acting on a Banach space X, and let λ be a nonzero scalar. Prove that

$$\dim \ker(T - \lambda I) < \infty.$$

(4) (15 marks) Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear map. Assume that

$$\varphi \circ T \in X^*,$$

for every $\varphi \in Y^*$. Prove that T is bounded.

(5) (20 marks) Let X be a normed space, and suppose that X is reflexive. Prove that X is separable if and only if X^* is separable.

(6) (20 marks) Let X be a normed space, and let $\varphi : X \to \mathbb{C}$ be a linear functional that is not identically zero. Prove that φ is not continuous if and only if $\ker \varphi$ is dense in X.